Electromagnetic Fields in Complex Media: A Time-Domain Analysis

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Abstract

This work is a review of recent results by the authors, presented by the second-named author at the one day meeting held on 5 June 2004 at the Department of Mathematics of the University of Ioannina in honour of Professors Sfikas and Staikos. It is based on results published in [3], [4]

1 Introduction

Until the 1960's, electromagnetic research focused on vacuum, metals, or dielectric media; sporadic attention to general electromagnetic media emerged rather slowly until the mid 1980's. Since then, the scene has dramatically altered: complex media electromagnetics is now a field of intensive theoretical and experimental research, with a wide range of technological applications. The integration of microwave circuits and components on aircraft and space vehicles requires very high performances in terms of speed and drag reduction: this is an important field of application of complex media. Further, complex media have useful properties when applied to microstrip waveguides and resonators, such as electronic frequency shifting, radar cross-section reduction, bandwidth enhancement, directivity improvement etc. Applications also arise in clinical medicine (turbid chiral media with nonchiral particulate inclusions are related with

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the concentration of blood glucose), planetary science (the atmosphere of Titan - the biggest satellite of Saturn - is expected to be characterised by a turbid chiral medium), physical chemistry (heterogeneous systems constituted of chiral particles in chiral fluids) and antennas industry (receiving and transmitting antennas in various types of sheaths).

Complex media are birefringent substances that respond to either electric or magnetic excitation with both electric and magnetic polarization. Such media have been known experimentally since the end of the 19th century (e.g. study of chirality by Pasteur) and, as mentioned above, find a wide range of applications from medicine to thin film technology. Under the name of complex media one includes a wide variety of different media such as chiral media, dispersive bianisotropic media etc. The understanding of the properties of such media, the differences from ordinary dielectrics, and their possible applications requires detailed mathematical modelling. The mathematical modelling of complex media is done through the modification of the constitutive relations for normal dielectrics. While for a normal dielectric the electric displacement $\mathbf{D}$ depends solely on the electric field $\mathbf{E}$, and the magnetic field $\mathbf{B}$ depends solely on the magnetic induction $\mathbf{H}$, in a chiral medium $\mathbf{D}$ and $\mathbf{B}$ depend on a combination of $\mathbf{E}$ and $\mathbf{H}$. In most cases of interest these constitutive laws are non-local relations containing $\mathbf{E}$ and $\mathbf{H}$. This is a common model for time-dispersive complex media. Also these constitutive laws may be either linear or nonlinear relations of the fields, corresponding to the modelling of linear or nonlinear complex media respectively.

Most of the mathematical work on complex media so far treats the time-harmonic case. Work in the time-domain while very interesting both from the mathematical point of view as well as from the point of view of applications is still relatively new. It is the aim of this paper to collect and review some recent results on the field, presented in [3], [4].

2 Linear bianisotropic dispersive media

This section is devoted to the mathematical modelling of an important and quite general class of complex media: linear bianisotropic dispersive media. These principles governing such media are based both on theoretical and experimental arguments. Our final goal is to derive the constitutive relations, that is to find the form of the operator C such that

$$\mathbf{d} = C\mathbf{u} \tag{1}$$

where $\mathbf{d} = (\mathbf{D}, \mathbf{B})^T$ and $\mathbf{u} = (\mathbf{E}, \mathbf{B})^T$. In particular, we suppose that

(P1) The medium reacts linearly to electromagnetic excitations.

(P2) The medium is causal with respect to electromagnetic excitations, that is if $\mathbf{u}(t) = \mathbf{0}$ for $t \in (-\infty, \tau]$ then $\mathbf{d}(t) = \mathbf{0}$ for $t \in (-\infty, \tau]$ too.

(P3) The electromagnetic quantities of the medium remain invariant to time translations, that is if to the electromagnetic field $\mathbf{u}(t)$ corresponds an electromagnetic displacement $\mathbf{d}(t)$, then to $\mathbf{u}_\tau(t) = \mathbf{u}(t-\tau)$, for arbitary $\tau > 0$, corresponds $\mathbf{d}_\tau(t) = \mathbf{d}(t-\tau)$.

(P4) The medium displays stability with respect to the excitations, in the sense that to "small" perturbations on $\mathbf{u}$ correspond "small" perturbation on $\mathbf{d}$

Let us discuss these assumptions at some depth. First of all, in the light of equation (1), we consider $\mathbf{u}$ as the excitation (cause) and $\mathbf{d}$ as the the result. (**P1**) is stated under the condition that the excitation is relatively small in amplitude, so a linear dependence between $\mathbf{d}$ and $\mathbf{u}$ can be assumed. (**P2**), as already remarked, is a causality requirement: we cannot know $\mathbf{d}$ before we start the observation of $\mathbf{u}$. (**P3**) says that the medium displays some sort of memory in the sense that the result $\mathbf{d}$, at each time instant t_0, depends on the past of the excitation $\mathbf{u}$, that is on the values $\mathbf{u}(t)$ for $t \in (-\infty, t_0]$.
In the sequel, we assume that $\mathbf{u} \in L^\infty_\omega(H)$. This means that the observation of $\mathbf{u}$ starts at $t = 0$ and we are not concerned about the past $t < 0$. Using (**P1**), (**P2**) and (**P4**) we have that C is a continous linear operator on $L^\infty_\omega(H)$. Taking into account and (**P4**), a form of (1), consistent with these assumption is

$$\mathbf{d} = A\mathbf{u} + K * \mathbf{u}$$

Here A and K are 6×6 matrices of the general block-form

$$A = \begin{bmatrix} \varepsilon & \xi \\ \zeta & \mu \end{bmatrix} \qquad K = \begin{bmatrix} \varepsilon_d & \xi_d \\ \zeta_d & \mu_d \end{bmatrix}$$

Matrix A models the instantaneous response of the medium to the excitations and is known in the literature as the *optical response*. The 3×3 matrices ε, ξ, ζ and μ have entries that are real, measurable, bounded functions of the spatial variable. There is a widely used classification of media with respect to these entries. In particular, one calls the medium

(M1) *homogeneous* if ε, ξ, ζ and μ are constant matrices,

(M2) *anisotropic* if $\xi = \zeta = \mathbf{0}$ and *isotropic* if, furthermore, ε and μ are proportional to identity matrix,

(M3) *bi-isotropic* if all ε, ξ, ζ and μ are all proportional to identity matrix and

(M4) *bianisotropic* in the general case.

Matrix K models the memory (time-dispesive) phenomena and is know as the *susceptibility kernel*. Its entries are real, sufficiently smooth, ω-exponentially bounded, causal functions of t. We also assume, for convenience, that $K(0) = \mathbf{0}$.
So, by inspection, we saw that a form of operator C, consistent with the assumptions (**P1**)–(**P4**), is

$$C = A + K* \tag{2}$$

We call (2) a *linear dispersive law.* Additional hypotheses can be made about the medium in order to model particular situations such as dissipation, reciprocity etc. These hypotheses lead to special forms of the matrices A and K. In this paper we suppose that the matrix A is coercive; this means that there exists a positive constant a so that for all $r \in \mathbb{C}^6$

$$Ar \cdot \overline{r} \geq a\,|\,r\,|^2 \tag{3}$$

Eventually, (3) ensures that A is symmetric matrix and therefore

$$\varepsilon = \varepsilon^T \qquad \mu = \mu^T \qquad \xi = \zeta^T$$

Furthermore, it is easy to see that A is invertible and A^{-1} is also symmetric.

3 Solvability of the problem

We are now in position to proceed to solving the considered problem. For convenience, we will treat the following version of the problem

$$\begin{cases} \dfrac{d}{dt}(A\mathbf{u} + K * \mathbf{u}) = \mathcal{M}\mathbf{u} + \mathbf{f} \\ \\ \mathbf{u}(0) = \mathbf{0} \end{cases} \tag{4}$$

Thus we assume homogeneous initial condition. The measurable, causal function f models the existent sources (currents) in the medium and presents the initial data of the problem. It is trivial to see that every problem with non-homogeneous initial condition $u_0 \in D(\mathcal{M})$ can be transformed in the form (4). We will call this problem, Problem I.

3.1 The optical response region

First we will assume the case where the dispersion phenomena are absent, that is the case $K = \mathbf{0}$. Then, using the invertibility of matrix A, (4) reduces to a usual Abstract Cauchy Problem

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} = \mathcal{M}_0\mathbf{u} + \mathbf{f}_0 \\ \\ \mathbf{u}(0) = \mathbf{0} \end{cases} \tag{5}$$

where $\mathcal{M}_0 = A^{-1}\mathcal{M}$ and $\mathrm{f}_0 = A^{-1}\mathbf{f}$. In the space H we consider the weighted inner product

$$\langle g_1, g_2 \rangle_A = \langle Ag_1, g_2 \rangle = \int_{\mathbb{R}^3} Ag_1 \cdot \overline{g_2}\, dr$$

Equipped with that, H becomes a Hilbert space. Thanks to (3), this inner product $\langle \cdot, \cdot \rangle_A$ is equivalent to the usual $\langle \cdot, \cdot \rangle$. It is well known that the Maxwell operator $\mathcal{M}$ is skew-adjoint with respect to $\langle \cdot, \cdot \rangle$. Then, by using the fact that A and A^{-1} are Hermitian, one can easily deduce that operator $\mathcal{M}_0$ is also skew-adjoint with respect to $\langle \cdot, \cdot \rangle_A$. Using Stone's Theorem, $\mathcal{M}_0$ is thus the infinitesimal generator of a unitary C_0-semigroup, say $\mathcal{G}$.

Let us know examine the problem using the vector-valued Laplace transform which is a generalization of the usual Laplace transform [1]. This approach is of formal nature and was proposed in the classic work of Hille and Yosida. Applying the Laplace transform to (5), we derive the *characteristic equation*

$$(\lambda\mathcal{I} - \mathcal{M}_0)\hat{\mathbf{u}}(\lambda) = \hat{\mathbf{f}}_0(\lambda) \tag{6}$$

The fact that $\mathcal{M}_0$ is skew-adjoint ensures that, for $\lambda \neq 0$, λ is a regular value of it, that is the resolvent $\mathcal{R}(\lambda) = (\lambda\mathcal{I} - \mathcal{M}_0)^{-1}$ is well defined on H and bounded. Indeed, the resolvent satisfies the inequality

$$\| \mathcal{R}(\lambda) \| \leq \frac{1}{\lambda} \tag{7}$$

After that, for each $\omega \geq 0$, the interval (ω, ∞) contains regular values of $\mathcal{M}_0$ and thus equation (6) can be written for $\lambda > \omega$

$$\hat{\mathbf{u}}(\lambda) = \mathcal{R}(\lambda)\hat{\mathbf{f}}_0(\lambda) \tag{8}$$

The Hille-Yosida theorem states that a closed, densely defined operator on a Hilbert space is the infinitesimal generator of a C_0-semigroup if and only if the resolvent is a member of the Widder class $W_\omega(\mathcal{B}(H))$. In this case the resolvent is the Laplace transform of the semigroup. Thus (8) takes the form

$$\hat{\mathbf{u}}(\lambda) = \hat{\mathcal{G}}(\lambda)\hat{\mathbf{f}}_0(\lambda)$$

and, after the inversion of the Laplace transform, we have

$$\mathbf{u}(t) = (\mathcal{G} * \mathbf{f}_0)(t) \tag{9}$$

So we have conclude to the explicit formula for the solution. Indeed, the following result holds

Theorem 3.1 *Let $\omega \geq 0$ and suppose $\mathbf{f} \in L^\infty_\omega(H)$ is continous with $\mathbf{f}(t) \in A\,[D(\mathcal{M})]$. Then (5) admits a unique classical (i.e. continously differentiable) solution $\mathbf{u} \in L^\infty_\omega(H)$, given by the formula (9). In addition, the problem is well posed.*

3.2 The general case

We proceed now to the general problem (4). Applying the Laplace transform, we take

$$\lambda A\hat{\mathbf{u}}(\lambda) + \lambda\hat{K}(\lambda)\hat{\mathbf{u}}(\lambda) = \mathcal{M}\hat{\mathbf{u}}(\lambda) + \hat{\mathbf{f}}(\lambda)$$

Again, multyplying from the left by A^{-1} and setting $\hat{K}_0 = A^{-1}K$

$$(\lambda\mathcal{I} - \mathcal{M}_0)\hat{\mathbf{u}}(\lambda) = -\lambda\hat{K}_0(\lambda)\hat{\mathbf{u}}(\lambda) + \hat{\mathbf{f}}_0(\lambda)$$

Thus we obtain the equation

$$\hat{\mathbf{u}}(\lambda) = -\lambda\mathcal{R}(\lambda)\hat{K}_0(\lambda)\hat{\mathbf{u}}(\lambda) + \mathcal{R}(\lambda)\hat{\mathbf{f}}_0(\lambda) \tag{10}$$

This is the form of a fixed-point problem for an affine operator, dependent on the parameter λ. The linear part of this operator is

$$\hat{T}(\lambda) = -\lambda\mathcal{R}(\lambda)\hat{K}_0(\lambda) \in \mathcal{B}(H)$$

Taking into account that $\hat{K}_0$ belongs to the Widder class, and using (7), we see that

$$\lim_{\lambda\to\infty} \left\| \hat{T}(\lambda) \right\| = 0$$

This means that for sufficiently large λ, say for $\lambda > \omega_0 \geq 0$, we have

$$\left\| \hat{T}(\lambda) \right\| < 1/2$$

namely $\hat{T}(\lambda)$ is a contraction. By the Banach Fixed Point Theorem (10) has, for these λ's, a unique solution $\hat{\mathbf{u}}(\lambda)$. This way a function $\hat{\mathbf{u}}$ is defined on the interval (ω_0, ∞) which eventually is continuous. We see that (10) can be written in terms of semigroups

$$\hat{\mathbf{u}}(\lambda) = \hat{\mathcal{G}}(\lambda)\left(\hat{\mathbf{f}}_0(\lambda) - \hat{L}(\lambda)\hat{\mathbf{u}}\right)$$

We have set, for convenience, $L = dK_0/dt$. Since $\hat{T}(\lambda) = -\hat{\mathcal{G}}(\lambda)\hat{L}(\lambda)$ is a contraction for $\lambda > \omega_0$, we know that $\mathcal{I} - \hat{T}(\lambda)$ is invertible and the following estimate holds

$$\left\| \left(\mathcal{I} - \hat{T}(\lambda)\right)^{-1} \right\| \leq \frac{1}{1 - \left\| \hat{T}(\lambda) \right\|} < 2 \tag{11}$$

Thus we take that

$$\hat{\mathbf{u}}(\lambda) = \left(\mathcal{I} - \hat{T}(\lambda)\right)^{-1} \hat{\mathcal{G}}(\lambda)\hat{\mathbf{f}}_0(\lambda) \tag{12}$$

Furthermore, we know that the sequence

$$\hat{\mathbf{u}}_n(\lambda) = \hat{\mathcal{G}}(\lambda)\left(\hat{\mathbf{f}}_0(\lambda) - \hat{L}(\lambda)\hat{\mathbf{u}}_{n-1}(\lambda)\right), \qquad \hat{\mathbf{u}}_0 \equiv \mathbf{0}$$

converges to the value $\hat{\mathbf{u}(\lambda)}$. In other words, $(\hat{\mathbf{u}}_n)$ converges pointwise to $\hat{\mathbf{u}}$ on (ω_0, ∞). From the above recursive formula, one easily computes

$$\hat{\mathbf{u}}_n(\lambda) = \left(\mathcal{I} - \hat{T}(\lambda)\right)^{-1}\left(\mathcal{I} - \hat{T}(\lambda)^n\right)\hat{\mathcal{G}}(\lambda)\hat{\mathbf{f}}_0(\lambda) \tag{13}$$

Theorem 3.2 *Let $\omega \geq \omega_0$ and suppose that $\mathbf{f} \in L^\infty_\omega(H)$ is continous with $\mathbf{f}(t) \in A\,[D(\mathcal{M})]$. Then the fixed-point problem, posed in space $L^\infty_\omega(H)$*

$$\mathbf{u} = \mathcal{G} * (\mathbf{f}_0 - L * \mathbf{u}) \tag{14}$$

has a unique solution $\mathbf{u}$ which is continously differentiable. This $\mathbf{u}$ is also the (classical) solution of the problem (4) and this problem is well posed. Finally, the following a priori *estimate holds*

$$\| \mathbf{u} \|_{\omega,\infty} \leq \frac{2}{a} \| \mathbf{f} \|_{\omega,\infty}$$

Proof. 1. EXISTENCE: It is true that $\mathcal{G}\hat{\mathbf{f}}_0 \in W_\omega(H)$. Using this fact, the estimate (11) and properties of the Laplace transform, we have from (12), in one hand, that $\hat{\mathbf{u}} \in W_\omega(H)$ (this means that $\hat{\mathbf{u}}$ is *really* a Laplace transform) and from (13), on the other hand, that $(\hat{\mathbf{u}}_n)$ is a Cauchy sequence in $W_\omega(H)$. So it converges in norm and its limit is $\hat{\mathbf{u}}$. After that, we may coniser equation (10) as an *identity* in $W_\omega(H)$. By inverting the Laplace transform, we take equation (14). Since the Laplace transform is an isometry (and thus a bicontinuous function), the sequence

$$\mathbf{u}_n = \mathcal{G} * (\mathbf{f}_0 - L * \mathbf{u}_{n-1}), \qquad \mathbf{u}_0 \equiv \mathbf{0} \tag{15}$$

converges in $L^\infty_\omega(H)$ to a function $\mathbf{u}$, the inverse of $\hat{\mathbf{u}}$ with respect to the Laplace transform. This $\mathbf{u}$ is a solution of (14). By induction and standard semigroup theory one can show that $\mathbf{u}_n$ is continuously differentiable and that $\mathbf{u}_n(t) \in D(\mathcal{M})$. Moreover, differentating the relation (14) we take exactly (4).
II. UNIQUENESS: Suppose $\mathbf{f} \equiv \mathbf{0}$. Then, by (12), $\hat{\mathbf{u}} \equiv \mathbf{0}$ is the unique solution of (10). The uniqueness follows from the injectivity of the Laplace transform.
III. STABILITY: Thanks to (12), the problem (10) is stable with respect to $\hat{\mathbf{f}}$. Using the continuity of of the inverse Laplace transform, it is clear that (14) is stable with respect to $\mathbf{f}$.
IV. ESTIMATE FOR THE SOLUTION: It follows immediately from (12), taking into account (3), (11) and the fact that the Laplace transform is an isometry. $\square$

Remark. On the proof of the theorem 3.2 it is evident that $\mathbf{u}_n$ is the solution of the Abstract Cauchy Problem

$$\begin{cases} \dfrac{d\mathbf{u}_n}{dt} = \mathcal{M}_0\mathbf{u}_n + (\mathbf{f}_0 - L * \mathbf{u}_{n-1}) \\ \\ \mathbf{u}(0) = \mathbf{0} \end{cases} \tag{16}$$

This means that the iterative scheme (16) converges to the solution of the original problem. This scheme can be further refined in order to be used for a numerical treatment of the problem, which avoids inversions of Laplace transforms.

4 The optical response approximation

In the previous section we treated the full nonlocal set of equations, modelling dispersive bi-anisotropic media, as far as solvability is concerned. Though the mathematical treatment of the full problem is feasible, in a number of important applications (for example in wave propagation or scattering problems) the full non-local problem may be cumbersome to handle. Thus, local approximations to the full problem have been proposed, that will keep the general features of complex media, without the mathematical complications introduced by the non-locality of the model. This section is devoted to the study of this problem in a special case, the case of linear dispersive chiral media. The special symmetries of such media facilitates the analysis.

In practice, a very common approximation scheme to the full constitutive relations for the medium is used, where essentially the convolution integrals are truncated to a Taylor series in the derivative of the fields. Using this expansion of the convolution integrals and the Maxwell constitutive relations we may obtain the so-called Drude-Born-Fedorov (DBF) constitutive relations for chiral media

$$\mathbf{D} = \epsilon(I + \beta curl)\mathbf{E}, \quad \mathbf{B} = \mu(I + \beta curl)\mathbf{H}$$

where β is the chirality measure, considered here as a parameter that will be chosen so that a criterion for optimality be satisfied. This approximation is usually called the *optical response approximation*. For such constitutive relations the equations for the fields become

$$
\begin{aligned}
curl\tilde{\mathbf{E}} &= -\frac{\partial}{\partial t}\left\{\mu(I+\beta curl)\tilde{\mathbf{H}}\right\} \\
curl\tilde{\mathbf{H}} &= \frac{\partial}{\partial t}\left\{\epsilon(I+\beta curl)\tilde{\mathbf{E}}\right\} \\
div\tilde{\mathbf{E}} &= 0, \quad div\tilde{\mathbf{H}} = 0
\end{aligned}
\tag{17}
$$

supplemented with the initial conditions

$$\tilde{\mathbf{E}}(x,0) = \mathbf{E}_0(x) \quad \tilde{\mathbf{H}}(x,0) = \mathbf{H}_0(x)$$

and the boundary conditions corresponding to the perfect conductor problem. This problem will be called hereafter Problem II. Its solvability is established in the following

Theorem 4.1 *Assuming the coercivity of matrix A and that the source terms have well defined Laplace transform, Problem II has a unique solution in $D[N]$ for sufficiently small β.*

The solution to Problem II is a commonly used approximation to the full solution of Problem I.

A very popular method of treating electromagnetic problems in the frequency domain is through the use of Beltrami fields. Another interesting approach to Problem II is through the use of Moses eigenfunctions [2]. These form a complete orthonormal basis for L^2 consisting of eigenfunctions of the curl operator.

Specifically, Moses [2] introduced three dimensional complex vectors $K(x,p;\lambda)$ with $x,p \in \mathbb{R}^3$ which satisfy,

$$curlK(x,p;\lambda) = \lambda\,|p|\,K(x,p;\lambda) \quad , \quad \lambda = 0, \pm 1. \tag{18}$$

That is, $K(x,p;\lambda)$ are eigenvectors of the *curl* operator and $\lambda\,|p|$ are the associated eigenvalues. These fields (that will be called Beltrami-Moses fields) satisfy some interesting orthogonality and completeness relations.

We may now define the fields

$$Q_\pm(x,t) = \{\mathbf{E} \pm i\eta\mathbf{H}\}(x,t) \quad , \quad \eta = \sqrt{\tfrac{\mu}{\epsilon}} \tag{19}$$

which implies

$$\mathbf{E}(x,t) = \tfrac{1}{2}\{Q_+ + Q_-\}(x,t) \quad , \quad \mathbf{H}(x,t) = \tfrac{1}{2i\eta}\{Q_+ - Q_-\}(x,t). \tag{20}$$

Using these fields we may proceed formally to rewrite Problem II in the following form

$$
\begin{aligned}
curl\,Q_\pm &= \pm i\sqrt{\mu\epsilon}\frac{\partial}{\partial t}\{(I+\beta curl)Q_\pm(x,t)\} \\
div\,Q_\pm(x,t) &= 0.
\end{aligned}
$$

The associated initial values are

$$Q_\pm(x,0) = \mathbf{E}_0(x) \pm i\eta\mathbf{H}_0(x) \tag{21}$$

Using these Beltrami-Moses fields as kernels for an integral transform we may define a generalized Fourier transform for vector functions $\psi(x,t)$, the Beltrami-Moses transform, as follows:

$$\hat{\psi}(p,t;\lambda) = \int \overline{K(x,p;\lambda)}\psi(x,t)dx.$$

The inverse transform is given by the formula

$$\psi(x,t) = \sum_{\lambda} \int K(x,p;\lambda)\hat{\psi}(p,t;\lambda)dp.$$

Expanding the fields $Q_{\pm}$ in terms of the Moses eigenfunctions and using the property that both these fields have to be divergence free we may reduce Problem II to a set of first order ordinary differential equations for the field amplitudes corresponding to $\lambda = \pm 1$. The electromagnetic fields may be obtained by inversion of the integral transform. This approach is related to the spectral approach to Problem II.

5 The error of the optical response approximation

Recall that $(\mathbf{E},\mathbf{H}), (\tilde{\mathbf{E}},\tilde{\mathbf{H}})$ are, respectively, the solutions of Problems I and II. We introduce a third problem the solution of which will furnish the error of the optical response approximation. So, let

$$w_E = \mathbf{E} - \tilde{\mathbf{E}}, \quad w_H = \mathbf{H} - \tilde{\mathbf{H}}.$$

After some elementary manipulations we find that the error of the optical response approximation satisfies the equations

$$\begin{aligned}
curl\, w_E &= -\frac{\partial}{\partial t}\left\{\mu w_H + \mu_1 \star w_H + \xi \star w_E + \mu_1 \star \tilde{\mathbf{H}} + \xi \star \tilde{\mathbf{E}} - \mu\beta curl \tilde{\mathbf{H}}\right\} \\
curl\, w_H &= \frac{\partial}{\partial t}\left\{\epsilon w_E + \epsilon_1 \star w_E + \zeta \star w_H + \epsilon_1 \star \tilde{\mathbf{E}} + \zeta \star \tilde{\mathbf{H}} - \epsilon\beta curl \tilde{\mathbf{E}}\right\} \\
curl\, \tilde{\mathbf{E}} &= -\frac{\partial}{\partial t}\left\{\mu(I + \beta curl)\tilde{\mathbf{H}}\right\} \\
curl \tilde{\mathbf{H}} &= \frac{\partial}{\partial t}\left\{\epsilon(I + \beta curl)\tilde{\mathbf{E}}\right\} \\
div\, w_E &= div\, w_H = div\, \tilde{\mathbf{E}} = div\, \tilde{\mathbf{H}} = 0
\end{aligned}$$

supplemented with the initial conditions

$$w_E(x,0) = 0, \; w_H(x,0) = 0, \; \tilde{\mathbf{E}}(x,0) = \mathbf{E}_0(x), \; \tilde{\mathbf{H}}(x,0) = \mathbf{H}_0(x).$$

This problem will be hereafter called Problem III. The solution of Problem III will furnish the error of the optical response approximation for a given solution $(\tilde{\mathbf{E}},\tilde{\mathbf{H}})$. Observe that the equations for the approximate fields are decoupled from the equations for the error.

A priori estimates are obtained on the solution of Problem III. This is done by reducing the error equations to the form of a Volterra equation of the second kind. By expanding the solution in Moses eigenfunctions we may rewrite the original system for the error in the compact form

$$A_1 w = \frac{d}{dt}\left\{A_2 w + A_3 \star w + S\right\}$$

where

$$w = \begin{pmatrix} w_{E,\lambda} \\ w_{H,\lambda} \end{pmatrix}, A_1 = \begin{pmatrix} \lambda \mid p \mid & 0 \\ 0 & \lambda \mid p \mid \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & -\mu \\ \epsilon & 0 \end{pmatrix}, A_3 = \begin{pmatrix} -\xi & -\mu_1(\tau) \\ \epsilon_1(\tau) & \zeta \end{pmatrix},$$

$$S = \begin{pmatrix} S_{1,\lambda} \\ S_{2,\lambda} \end{pmatrix} = \begin{pmatrix} -\mu_1 \star \bar{H}_\lambda - \xi \star \bar{E}_\lambda + \lambda\beta\mu \mid p \mid \bar{H}_\lambda \\ \epsilon_1 \star \bar{E}_\lambda + \zeta \star \bar{H}_\lambda - \lambda\beta\epsilon \mid p \mid \bar{E}_\lambda \end{pmatrix}.$$

Now integrate once over time to rewrite the equation for the error in the following form

$$w = \phi \star w + g \tag{22}$$

where

$$\phi = A_2^{-1}(A_1 - A_3), \quad g = -A_2^{-1} S.$$

For the specific system we study here we have that

$$\phi = \begin{pmatrix} -\frac{\epsilon_1(\tau)}{\epsilon} & \frac{\lambda|p|-\zeta}{\epsilon} \\ -\frac{\lambda|p|+\xi}{\mu} & -\frac{\mu_1(\tau)}{\mu} \end{pmatrix}, \quad g = \begin{pmatrix} \frac{\epsilon_1}{\epsilon} \star \bar{E}_\lambda + \frac{\zeta}{\epsilon} \star \bar{H}_\lambda - \lambda\beta \mid p \mid \bar{E}_\lambda \\ \frac{\mu_1}{\mu} \star \bar{H}_\lambda + \frac{\xi}{\mu} \star \bar{E}_\lambda - \lambda\beta \mid p \mid \bar{H}_\lambda \end{pmatrix}.$$

This matrix Volterra equation will be used to obtain *a priori* estimates for the error of the optical response approximation in terms of the Moses transformed fields. The following holds [3].

Theorem 5.1 *Let*

$$\Psi(t) := (1 - 2\sup_{i,j} \| \phi_{ij} \|_{L_1(0,t)})^{-1} > 0.$$

Then, the solution of (22) satisfies the following a priori error bound

$$\sup_i \| w_i \|_{L_p(0,t)} \leq \Psi(t) \sup_i \| g_i \|_{L_p(0,t)} .$$

It is interesting to notice that an alternative method of obtaining a priori bounds can be developed using the Gronwall inequality. Indeed, in this manner we can readily obtain the following result.

Theorem 5.2 *Suppose $\epsilon > 0$, $\mu > 0$, $\xi > 0$ and $\zeta > 0$ and that $\mid p \mid \leq min(\xi, \zeta)$. Assuming that the functions ϕ_{ij} are bounded, we obtain*

$$\sup_t \mid w_1(t) + w_2(t) \mid \leq \sup_t \mid g_1(t) + g_2(t) \mid .$$

The above estimate provides us with a way to minimize the error of the optical response approximation. One way to do this, is by minimizing the upper bound $\sup_i \| g_i \|_{L_r(0,t)}$. This amounts to choosing the value of β so as to minimize the integrals

$$\| g_1 \|_{L_r(0,t)} = \left\{ \int_0^t \left| \frac{\epsilon_1}{\epsilon} \star \bar{E}_\lambda + \frac{\zeta}{\epsilon} \star \bar{H}_\lambda - \lambda\beta \mid p \mid \bar{E}_\lambda \right|^r dt' \right\}^{1/r}$$

$$\| g_2 \|_{L_r(0,t)} = \left\{ \int_0^t \left| \frac{\mu_1}{\mu} \star \bar{H}_\lambda + \frac{\xi}{\mu} \star \bar{E}_\lambda - \lambda\beta \mid p \mid \bar{H}_\lambda \right|^r dt' \right\}^{1/r} .$$

A series of other results were obtained for each p using the expansion of Problem III in Moses eigenfunctions. This approach allows us to find exact forms for the Laplace transform of the error for specified wavenumbers. Numerical techniques can thus be used for the inversion of the Laplace transform and the retrieval of the time dependence of the error term.

An estimate of the error in the spatial variables rather than in terms of the wavenumbers can be obtained in the following way. Adopting the notation of Section 2, the equation for the error may written in the form

$$Lw = \frac{\partial}{\partial t}(Aw + K \star w + \Phi)$$

where Φ is a source term which is related to the solutions of the optical response equation $\bar{\mathbf{H}}$ and $\bar{\mathbf{E}}$. Multiplying by w, integrating over space and using the properties of the operator L we obtain

$$\frac{1}{2}\frac{d}{dt}(\| w \|^2) + < \frac{d}{dt}(K \star w), w > + < \Phi, w > .$$

Under the assumption that the convolution kernel is such that

$$K_1 Aw \leq \frac{d}{dt}(K \star w) \leq K_2 Aw$$

we obtain

$$\frac{d}{dt} \| w \|^2 + K_1 \| w \|^2 \leq \mid \Phi \mid\mid\mid w \|$$

from which by use of the Gronwall inequality we may obtain a priori bounds for the error. Similar bounds may be obtained by slight modification of the conditions on the kernels.

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